On the Mass Distribution of Planetesimals in the Early Runaway Stage

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Abstract

We derive the stationary distribution of the mass of planetesimals from the coagulation equation under the assumption of energy equipartition. In the three-dimensional case, we found that the stationary solution is expressed as $n(m) \propto m^{-8/3}$, where n is the surface number density and m is the mass of the planetesimals. This solution is in excellent agreement with the result of direct N-body simulations and the numerical integration of the coagulation equation, for the range of mass which contains most of the total mass of the planetesimals (10^{22}g — 10^{25}g). In the two-dimensional case, the stationary solution exists but cannot be realized in a finite time. This difference is the direct consequence of the fact that runaway growth takes place in three dimensions, but not in two dimensions.

Key words: Planetary Formation, Planetary Accretion, Runaway Growth, Planetesimals

1 Introduction

It is generally accepted that the terrestrial planets and the cores of the Jovian planets were formed through the accretion of many small bodies called planetesimals. The accretion process is rather complex, since several competing physical mechanisms are working simultaneously (see *e.g.* Lissauer 1993). In particular, the interplay of the velocity distribution and the mass distribution is quite complex.

There have been several numerical studies of the accretion process, in which the coupling between the evolution of the velocity distribution and the mass distribution was taken into account. The importance of the coupling was first

pointed out by Greenberg *et al.* (1978). Wetherill and Stewart (1989) used the "particle-in-a-box" approach with various assumptions for the velocity distribution to study the coupling. They found "runaway growth" of massive planetesimals in the simulations which included the effect of the energy equipartition.

Kokubo and Ida (1996) performed full three-dimensional N-body simulations of the planetary accretion process, and confirmed that runaway growth takes place. They also found that the mass distribution first relaxes to the power-law form

$$n(m) \propto m^{\alpha}, \tag{1}$$

with $\alpha \simeq -2.5$, for the mass range covered by their calculation ($10^{23}\mathrm{g} < m < 10^{25}\mathrm{g}$). Here, $n(m)dm$ gives the surface number density of planetesimals with mass between m and $m + dm$. Earlier studies based on more approximate theory (Wetherill and Stewart 1989, 1993, Barge and Pellat 1991, 1993) also showed similar results for that mass range. On the other hand, in the case of two-dimensional simulations, no simple power-law mass distribution was realized.

It should be noted that this range of mass contains most of the mass when the runaway takes place. For masses less than 10^{22}g, planetesimals exhibit orderly growth, because the condition for the runaway is not satisfied (*e.g.* Ohtsuki and Ida 1990).

In addition, this power-law mass distribution is realized almost immediately once the runaway starts. Therefore, the same mechanism which drives the runaway growth should be responsible for this power law mass distribution. However, no theoretical explanation has been proposed yet.

In this paper, we present an exact stationary solution of the coagulation equation which explains why the power-law mass distribution is realized in the early stage of the accretion process. In section 2, we derive the exact stationary solution of the coagulation equation. In section 3, we present a more intuitive description of the mechanism of the formation of the power-law distribution. In section 4, we discuss the implications of our result.

2 Theory

The presence of the power-law distribution implies that the evolution of the mass distribution function $n(m)$ follows an approximately self-similar solution. Instead of trying to obtain the self-similar solution, however, here we try to

find a stationary solution, with the supply of the planetesimals at the low-mass end and the removal of the massive planetesimals at the high-mass end. This is a usual exercise in studying the distribution function analytically.

The change in the surface number density $n(m)$ of planetesimals with mass m is expressed as

$$\frac{\partial n}{\partial t} = f_+ - f_-, \tag{2}$$

where f_+ and f_- are the incoming and outgoing fluxes. They are defined as

$$f_+ = \int_0^{m/2} P(m', m-m')n(m')n(m-m')dm', \tag{3}$$

$$f_- = n(m) \int_0^{\infty} P(m, m')n(m')dm'. \tag{4}$$

Here, $P(m, m')$ is the probability that two planetesimals of masses m and m' collide in a unit time. This is the classical Smoluchowski-Safronov coagulation equation (Safronov 1969) integrated over the velocity distribution. The integration over velocity is justified on the ground that the relaxation timescale is shorter than the coagulation timescale.

Note that we assume that when two planetesimals collide they always stick together (perfect accretion). This assumption is okay for planetesimals since the velocity dispersion is lower than the escape velocity.

The collision probability $P(m, m')$ is expressed roughly as

$$P(m, m') = \frac{1}{\max(H, H')} \pi (r + r')^2 \left(1 + \frac{v_{\rm esc}^2}{v_{\rm rel}^2}\right) v_{\rm rel}, \tag{5}$$

where H is the scale height, r is the radius of a planetesimal, $v_{\rm esc}$ is the escape velocity given by

$$\frac{1}{2} v_{\rm esc}^2 = G \frac{m + m'}{r + r'}, \tag{6}$$

and $v_{\rm rel}$ is the average relative velocity given by

$$v_{\rm rel}^2 = v^2 + v'^2. \tag{7}$$

Here G is the gravitational constant. Note that H', r', and v' denote corresponding quantities for the planetesimal of mass m'.

In the following, we assume that the velocity distribution of the planetesimals satisfies the thermal equilibrium

$$H \propto v \propto m^{-1/2}, \tag{8}$$

and that the velocity is in the extreme gravitational focusing region,

$$v_{\rm esc} >> v_{\rm rel}. \tag{9}$$

Here we assumed that the two-body formalism can be used to obtain the collision cross section. This assumption is valid since the velocity dispersion is in the "dispersion dominant" regime while the runaway growth takes place.

Our goal here is to obtain the mass distribution function $n(m)$ for which $\partial n/\partial t = 0$ for all values of m. Before trying to obtain the solution, let us first investigate the characteristics of the collision probability P. From equations (5) to (8), we can find the behavior of P in two limiting cases

$$P \rightarrow \begin{cases} m' m^{4/3} & (m' << m) \\ m m'^{4/3} & (m << m'). \end{cases} \tag{10}$$

To put it in a slightly different way, we can express P as

$$P = m^{\beta} h(m'/m), \tag{11}$$

with $\beta = 7/3$. Here $h(\theta)$ is a function which has the asymptotic behavior of:

$$h(\theta) \rightarrow \begin{cases} \theta & (\theta << 1) \\ \theta^{4/3} & (\theta >> 1). \end{cases} \tag{12}$$

Note that equation (11) combined with equation (12) is equivalent to equation (10), but still is exact as long as we retain the function h.

For the power-law mass distribution of equation (1), equation (10) implies that both f_+ and f_- diverge in the low-mass end if $\alpha \leq -2$, and f_- diverges also at the high-mass end if $\alpha > -7/3$. In other words, for any value of α, either of f_+ or f_- diverges. In particular, for the experimental result of $\alpha \simeq -2.5$, both diverge at the low-mass end.

The divergence at the low-mass end is, however, not a physical reality, but a mathematical artifact caused by the inadequate form of equations (3) and

(4). In the limit of $m' \to 0$, the apparent flux expressed in equations (3) and (4) diverges. However, the mass of particles which originally had a mass of m or $m - m'$ does not change in the limit of $m' \to 0$. In other words, there is an infinite flux, in between the two mass bins with infinitesimal separation. The net effect of the product of the infinite and the infinitesimal must be carefully examined.

A convenient way to avoid this difficulty of the apparent divergence of f is to introduce the "mass flux" F, in such a way that its partial derivative in mass space gives the net change of the distribution function as

$$m\frac{\partial n}{\partial t} = -\frac{\partial F}{\partial m}. \tag{13}$$

Tanaka *et al.* (1996) used this form to study the collision cascade process. The formal derivation here is essentially the same as theirs. Since equation (13) implies that F is the integral of $m(f_+ - f_-)$, one might think it should behave in exactly the same way. However, as we'll see shortly, F does not diverge at the low-mass end, even though f_+ and f_- diverge. This is because the contribution of the low-mass end to the mass flux F is small. As we stated earlier, the change of mass vanishes at the low-mass end. Thus, even though f_+ and f_- diverge, their contribution to F vanishes at the low-mass end.

The mass flux F is calculated as

$$F(m) = \int_0^m dm_1 \int_{m-m_1}^{\infty} dm_2 m_1 P(m_1, m_2) n(m_1) n(m_2). \tag{14}$$

It is straightforward to prove that the combination of equations (13) and (14) is formally equivalent to equations (2) through (4).

As shown by Tanaka *et al.* (1996), for a collision rate of the form of equation (11), F is reduced to

$$F = m^{2\alpha+\beta+3} \int_0^1 dx \int_{1/x-1}^{\infty} dy x^{2\alpha+\beta+2} y^{\alpha} h(y), \tag{15}$$

where $x = m_1/m$ and $y = m_2/m_1$.

The stationary solution corresponds to the case $F = \text{const.}$, which is realized when

$$2\alpha + \beta + 3 = 0. \tag{16}$$

As stressed by Tanaka *et al.* (1996) this result does not depend on the functional form of $h(\theta)$, as far as $P(m, m')$ can be expressed in the form of equation (11).

The double integral in equation (15) should have a finite value. To determine if it is the case or not, it is more convenient to rewrite the double integral in a slightly different form as

$$F = m^{2\alpha+\beta+3} \int_0^1 dx \int_{1-x}^{\infty} dz x^{\alpha+\beta+1} z^{\alpha} h(z/x), \tag{17}$$

where $z = m_2/m$.

Without losing generality, we can assume that h has the following asymptotic expression

$$h(\theta) \rightarrow \begin{cases} \theta^{\gamma} & (\theta << 1) \\ \theta^{\beta-\gamma} & (\theta >> 1). \end{cases} \tag{18}$$

Since $P(m, m')$ is symmetric, the limiting behavior of h in either limit determines the behavior in the other limit. Equation (12) corresponds to the case of $\gamma = 1$.

The condition that F is finite is given by

$$\begin{aligned} \alpha + \beta - \gamma &< -1, \\ \alpha + \gamma &> -2. \end{aligned} \tag{19}$$

The first inequality comes from the condition that the integrand should approach zero faster than $1/z$ in the limit of $z \rightarrow \infty$. The second comes from the condition that the integrand should not diverge as $1/x$ or faster in the limit of $x \rightarrow 0$. This inequality also guarantees that the integrand does not diverge as $1/z^2$ or faster in the limit of $z \rightarrow 0$. Note that it can diverge faster than $1/z$, since the range of the integration over x is proportional to z.

The criterion (19) is different from what is shown in the Appendix of Tanaka *et al.* (1996). Their derivation did not incorporate the effect of the power of h on the convergence criterion correctly.

For $\beta = 7/3$, we have $\alpha = -8/3$. As noted above, $\gamma = 1$. This set of values satisfies the convergence criteria. Thus, we found a stationary solution of the coagulation equation expressed as $n(m) \propto m^{-8/3}$. This is in quite good agreement with the numerical results obtained by N-body or Fokker-Planck calculations.

3 Interpretation

Though the result obtained in the previous section is simple and in good agreement with the experimental result, the mechanism which determines the value of α is not at all clear. In this section, we try to determine α with more approximate approaches and to obtain qualitative understanding. First, we obtain the growth rate and growth mode of a typical planetesimal, using the collision probability obtained in section 2, and then derive the stationary distribution from them.

The growth rate of a planetesimal is obtained by integrating equation (5) over the range $(0, m)$ as (*e.g.* Safronov 1969, Barge and Pellat 1991)

$$\frac{dm}{dt} = \int_0^m n(m') P m' dm'. \tag{20}$$

By substituting equations (5) through (8) into equation (20) and neglecting the dependence on m' of $v_{\rm esc}$ and r and that on m of $v_{\rm rel}$, we have

$$\frac{dm}{dt} \propto m^{4/3} \int_0^m n(m') m'^2 dm'. \tag{21}$$

For the power-law mass distribution of equation (1), equation (21) is reduced to

$$\frac{dm}{dt} \propto \begin{cases} m^{\alpha+13/3} & (\alpha > -3) \\ m_0^{\alpha+3} m^{4/3} & (\alpha < -3), \end{cases} \tag{22}$$

where m_0 is the cut-off mass of the distribution function at the low-mass end. These two cases correspond to two different modes of accretion. If $\alpha < -3$, the main mode of the growth is to eat small planetesimals, since the growth rate is determined by the low end of the integration over mass. If $\alpha > -3$, on the other hand, the growth is driven by the collision with particles with similar masses. In other words, the growth is hierarchical.

Both the numerical results and the result obtained in the previous section suggest that $\alpha \sim -8/3$. So we consider the case $\alpha > -3$. The stationary state is realized if the mass flux does not depend on either the time or the mass, which is expressed as

$$nm\frac{dm}{dt} = \text{const.} \tag{23}$$

Note that here the condition of the stationary distribution is the constant flux of the mass nm, and not the constant flux of the number of planetesimals n. This is because the dominant mode of the growth of the planetesimals is the hierarchical growth. The number of planetesimals would become smaller as they grow, since growth is driven by collisions of planetesimals of similar sizes.

If collisions are dominantly hierarchical, strictly speaking it does not make sense to use the growth equation expressed as the differential equation (20), since the change of the mass of planetesimals is not continuous. However, in this case we can use the mass doubling timescale

$$t_{\rm double} = m(dm/dt)^{-1}, \tag{24}$$

and consider the discrete mass bins with the ranges $[m_0, 2m_0)$, $[2m_0, 4m_0)$, $[4m_0, 8m_0)$, ... $[2^k m_0, 2^{k+1} m_0)$, ... In this picture, collisions take place only between particles in the same mass bins. The product of the collision moves to the next mass bin.

In this case, the stationary state is realized when the incoming and outgoing flux for each mass bin are equal. For mass bin k, the total mass of planetesimals lost per unit time is

$$M_{-,k} = M_k t_{{\rm double},k}^{-1}, \tag{25}$$

where the subscript k denotes the mass bin k and M_k is the total mass of planetesimals in the mass bin k. Note that $M_k = m_k^2 n$, since the width of the mass bin is proportional to m itself.

The incoming mass to bin k is what is lost from bin $k-1$. Thus, the condition for the stationary state is that the outgoing mass is the same for all mass bins,

$$M_{-,k} = {\rm const.} \tag{26}$$

Equations (24) through (26) leads to equation (23) as the condition for the stationary state. Thus, whether we assume continuous growth of the mass or discontinuous jumps in the mass, the condition for the stationary state is the same.

From equation (1), (22), and (23), we obtain

$$m^{2\alpha+16/3} = {\rm const.} \tag{27}$$

and therefore

$$\alpha = -\frac{8}{3}. \tag{28}$$

This value is the same as we obtained in section 2.

If we assume a constant flux of the number of planetesimals, we obtain $\alpha = -13/6$, which is significantly smaller.

The reason why α has a large negative value is now clear. As can be seen in equation (22), the growth rate of planetesimals has a strong positive dependence on their mass. This is, of course, why the runaway growth occurs (Wetherill and Stewart 1989). This strong dependence requires that the number of planetesimal goes down quickly as its mass increases, if a stationary distribution is to be realized. The strong dependence of the growth rate on the mass comes from the size, scale height and the effect of gravitational focusing, all of which enhance the growth rate for larger mass.

From equations (22) and (28), we obtain

$$\frac{1}{m}\frac{dm}{dt} \propto m^{2/3}, \tag{29}$$

which implies the mass of a planetesimal can reach infinity in a finite time. Of course, this cannot happen in reality, since the number of planetesimals with this infinite mass would go below unity. In addition, at a certain point the most massive particle grows so massive that its effect dominates the velocity dispersion of less massive particles, resulting in the slowing down of the growth (Ida and Makino 1993). This slowing down causes the massive planetesimals of similar mass to be formed in roughly equal orbital separation. Thus, these massive particles cannot collide with each other (Kokubo and Ida 1995). Nevertheless, equation (29) implies that the stationary distribution of the mass for a wide range of masses can be realized in a short time.

It should be noted that the growth rate of equation (29) is larger than what has been used for the timescale estimate for the runaway stage (see . *e.g.*, Ida and Makino 1993). In previous estimates, uniform background distribution of planetesimals was assumed. Under this assumption, the power of m in equation (29) becomes 1/3.

In the case of the two-dimensional space, equation (5) becomes

$$P(m, m') = 2(r + r')\sqrt{\left(1 + \frac{v_{\rm esc}^2}{v_{\rm rel}^2}\right)}v_{\rm rel}, \tag{30}$$

which gives $\beta = 2/3$. Therefore. from equation (16) we obtain

$$\alpha = -\frac{11}{6}. \tag{31}$$

However, with this formula, the total mass diverges if the high-mass end is taken infinite. Therefore, for the total mass to conserve, there must be a cutoff at the high-mass end, and no stationary solution is possible. In other words, the growth would be an orderly one.

The value of α for two-dimensional case coincides with that for collisional cascade (Tanaka *et al.* 1996). This is simply because the value of β is the same.

Note that the existence of the stationary solution is directly related to the occurrence of the runaway growth. The runaway implies that the heavy particles can become heavier by themselves, without being affected by lighter particles. Thus, the power-law behavior is observed because there is no characteristic scale. In the case of the two-dimensional simulation, the evolution of the heaviest particle is slower than that of lighter particles, and therefore all particles tend to have a mass similar to that of the heaviest particle. The power-law behavior is not observed in this case.

4 Discussion and summary

4.1 Mass function for smaller masses

Up to here, we discussed the mass function of planetesimals for the mass range higher than 10^{22}g, where the runaway growth takes place. Here we consider the smaller mass range, where the evolution is more complex.

Wetherill and Stewart (1993) performed extensive simulations using a particle-in-a-box approach, which included both the coagulation and fragmentation, and obtained the mass function. Their result did not depend on the details of the calculation and could be summarized as follows: For the mass range 10^{21-24}, the power index is close to -2.5, in good agreement with our analytical result. For smaller mass, it seems that the power index is approximately -1.7.

For this low-mass region, we can assume that evolution in their simulation is dominated by the fragmentation process. The formalism of Tanaka *et al.* (1996) can also be used for this case. Here, however, the collision rate P has a power index different from that for the runaway growth region, because the equipartition of energy is not established and gravitational focusing is not effective.

Thus, relative velocity is independent of the mass and collision cross section is geometrical. Therefore we have $\beta = 2/3$ and $\alpha = -11/6$. This result is in fair agreement with the numerical result of Wetherill and Stewart (1993), for the intermediate mass range ($10^{10} - 10^{15}$g). This value of α is the same as that for the collisional cascade (Tanaka *et al.* 1996), because in both cases the collision cross section is geometrical.

For even smaller masses, they obtained a somewhat shallower slope. This is the natural result of the positive dependence of the velocity dispersion on the mass due to strong gas drag.

4.2 Summary

Using an exact analytic treatment, we obtained the power-law mass distribution of planetesimals which has been found in numerical simulations of runaway growth. The mechanism through which the power-law distribution is realized is directly related to the occurrence of runaway growth, where heavier planetesimals grow faster than lighter planetesimals.

Our treatment of runaway growth is different from those in previous theoretical works (see, *e.g.*, Ohtsuki and Ida 1990, Ida and Makino 1992a, 1992b), where the growth rate was obtained for different planetesimals in the background distribution of single-mass planetesimals. In those analyses, the difference in the growth rates can come only from the difference in the mass of the planetesimals under consideration. In other words, previous works dealt with the growth of the small perturbation of the mass from the uniform distribution (linear stability analysis).

In the present work, we investigated the evolution of the mass distribution function which is far from uniform, taking into account the non-uniformity of the background distribution as well. As a result, we were able to derive the stationary distribution, which is consistent with the result of numerical experiments.

Our result suggests that the power-law mass distribution of planetesimals is a universal law which is realized in the early stage of runaway growth. Since the evolution of the mass function in the late runaway stage is quite slow (Lissauer 1993, Kokubo and Ida 1996), we can conclude that the distribution of the mass of planetesimals follows this power law for most of the planet formation period. The mass distribution for smaller masses (less than 10^{21}g) is expressed by a different power law, since in this regime energy equipartition is not established.

Acknowledgments

We are most grateful to Hidekazu Tanaka, for critical comments on the early version of this paper and for extremely valuable discussions on the relation between present work and his work. We also thank Shigeru Ida for useful discussions and Gren Stewart and Luke Dones, the reviewer, for productive comments on the manuscript.

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