Thermal Relaxation in One-Dimensional Self-Gravitating Systems

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Abstract

In this paper, we study the thermal relaxation in the one-dimensional self-gravitating system, or the so-called sheet model. According to the standard argument, the thermal relaxation time of the system is around Nt_c, where N is the number of sheets and t_c is the crossing time. It has been claimed that the system does not reach the thermal equilibrium in this thermal relaxation timescale, and that it takes much longer time for the system to reach true thermal equilibrium.

We demonstrate that this behavior is explained simply by the fact that the relaxation time is long. The relaxation time of sheets with average binding energy is $\sim 20Nt_c$, and that of sheets with high energy can exceed $1000Nt_c$. Thus, one needs to take the average over the relaxation timescale of high-energy sheets, if one wants to look at the thermal characteristic of these high energy sheets.

1 Introduction

The one-dimensional self-gravitating many-body system was originally discussed mainly as a simple toy model to understand the violent relaxation, [4] because the thermal relaxation timescale of its discrete realization, the sheet model, was believed to be long. Until 1980s, it had been generally accepted that the thermal relaxation time of the system of N equal-mass sheets is of the order of N^2t_c, where t_c is the crossing time of the system.

However, by means of numerical simulation Luwel *et al.*[3] have demonstrated that the relaxation time is of the order of Nt_c. Reidel and Miller [6, 7] reached a similar conclusion, though they reported the presence of systems which apparently did not relax for much longer timescale.

In a series of papers, Tsuchiya *et al.* [10, 11, 12] have studied the thermal relaxation process of one-dimensional self-gravitating systems in detail, by means of the numerical integration over very long timescale (some of their experiments covered $5 \times 10^8 t_c$). They claimed that the thermal relaxation of the sheet model proceeds in a highly complex manner. In the "microscopic relaxation timescale" of Nt_c, each sheet forgets its initial condition, and the system is well mixed. However, according to them, the system does not really reach the thermal equilibrium in this timescale, and the distribution function remains different from that of the isothermal state. They called this state a quasiequilibrium

By pursuing the time integration for much longer timescale, Tsuchiya *et al.* [12] found that the system exhibits the transition from one quasiequilibrium to another, and they claimed that the thermal equilibrium is only realized by averaging over the timescale longer than the timescale of these transitions. Thus, they argued that there exists the timescale

for "macroscopic" relaxation, which is much longer than the usual thermal relaxation (what they called "microscopic relaxation").

In this paper, we try to examine the nature of this "macroscopic" relaxation of the one-dimensional sheet model. In section 2, we describe the numerical model. In section 3, we present the result of the measurement of the relaxation time. It is shown that the relaxation time, defined as the timescale in which individual sheets change their energies, depends very strongly on the energy itself, and is very long for high energy sheets. This strong dependence of the relaxation timescale on the energy naturally explains the apparent "transient" phenomena observed by Tsuchiya *et al.*[12] Section 4 discusses the implication and relevance of our results.

2 The Model

2.1 Sheet model

The Hamiltonian of the sheet model is given by

$$H = \frac{m}{2}\sum_{i=1}^{N} v_i^2 + 2\pi G m^2 \sum_{i<j} |x_i - x_j|, \tag{1}$$

where x_i and v_i are the position and velocity of sheet i, m is the mass of the sheets, N is the number of the sheets and G is the gravitational constant. The crossing time is defined as

$$t_c = \frac{1}{4\pi G M}\sqrt{\frac{4E}{M}}, \tag{2}$$

where $M = mN$ is the total mass of the system. Following Tsuchiya *et al.*[12] and others, we use the system of units in which $M = 4E = 4\pi G = 1$. In this system, $t_c = 1$.

A unique nature of the one-dimensional gravitational system is that there exists the thermal equilibrium, unlike its counterpart in three dimensions. Rybicki [8] obtained the distribution function

$$f(\varepsilon) = \frac{1}{8}\left(\frac{1}{2\pi}\right)^{1/2}\left(\frac{3M}{2E}\right)^{3/2}\exp\left(-\frac{3M}{2E}\varepsilon\right), \tag{3}$$

where ε is the specific binding energy defined as

$$\varepsilon = \frac{v^2}{2} + \Psi(x) - \Psi(0). \tag{4}$$

Here, $\Psi(x)$ is the specific potential energy. This distribution function satisfies the relation

$$\exp\left(-\frac{3M}{2E}\varepsilon\right) = \mathrm{sech}^2\left(\frac{3x}{8E}\right). \tag{5}$$

We performed the time integration of the system with $N = 16$, 32, 64, 128 and 256. For all systems, the initial condition is a water-bag with the aspect ratio $x_{max}/v_{max} = 2.5$.

2.2 Numerical method

The important character of the sheet model is that one can calculate the exact orbit of each sheet until two sheets cross each other. Thus, we can integrate the evolution of the system precisely (except for the round-off error). This may sound like a great advantage, compared to the systems in higher dimensions whose orbits can be calculated only numerically. Instead

of numerically integrating the orbit of each sheet, we can calculate the exact orbit for any sheet, until it collides with the neighboring sheet. Thus, by arranging the pairs using heap, we can handle each collision in $\log N$ calculation cost.

Note, however, that typically each sheet collides with all other sheets in one crossing time. Thus, the calculation cost is $O(N^2 \log N)$ per crossing time. Our simulation with $N = 64$ for $2 \times 10^7 t_c$ took 8 hours on a VT-Alpha workstation with DEC Alpha 21164A CPU running at 533 MHz. For this run, the total energy of the system was conserved better than 3×10^{-12}.

3 Results

3.1 Approach to the thermal equilibrium

Figure 1 shows the time-averaged energy distribution function $N(\varepsilon)$, for different time periods and number of sheets. In all figures, the thin solid curve is the energy distribution of the isothermal distribution function of equation (3). What we see is quite clear. As we make the time interval longer, the time-averaged distribution function approaches to the isothermal distribution. Thus, the numerical result suggests the system is ergodic. However, it also shows that the time needed to populate the high-energy region is very long. The sampling time interval is 128 time units for $N = 16$, and 512 time units for $N = 64$ and 128. Thus, in the case of $T = 2^{18}$ and $N = 16$ (dash-dotted curve in figure 1a), total number of sample points is $2^{15} = 32768$.

If we can assume that the sample points are uncorrelated, the possibility that no sample exceeds energy level ε_0 is given simply by

$$P(\varepsilon_0, N) = [1 - P(\varepsilon < \varepsilon_0)]^n, \tag{6}$$

where

$$P(\varepsilon < \varepsilon_0) = \int_0^{\varepsilon_0} N(\varepsilon) d\varepsilon, \tag{7}$$

and n is the number of sample points. Figure 2 shows $1 - P(\varepsilon)$ as a function of ε. For $\varepsilon = 1.25$, $P(\varepsilon) = 0.996$, and therefore the probability that none of 32768 samples does not exceed $\varepsilon = 1.25$ is practically zero ($< e^{-100}$). In other words, the numerical result seems to suggest that the system is not in the thermally relaxed state even after 2×10^5 crossing times.

Of course, this result is not surprising if the relaxation time is long. Samples taken with the time interval shorter than the relaxation time have a strong correlation, and therefore the effective number of freedom can be smaller than n. Roughly speaking, if the relaxation time is longer than 10^4, our numerical result is consistent with the assumption that the system is in the thermal equilibrium. In the next subsection, we investigate the relaxation time itself.

3.2 Relaxation timescale

We measured the following quantities:

$$D_1 = \frac{< \varepsilon_i(t_0) - \varepsilon_i(t_0 + \Delta t) >}{\Delta t}, \tag{8}$$

$$D_2 = \frac{< [\varepsilon_i(t_0) - \varepsilon_i(t_0 + \Delta t)]^2 >}{\Delta t}. \tag{9}$$

These quantities correspond to the coefficients of the first and second-order terms in the Fokker-Planck equation for the distribution function, and have been used as the measure

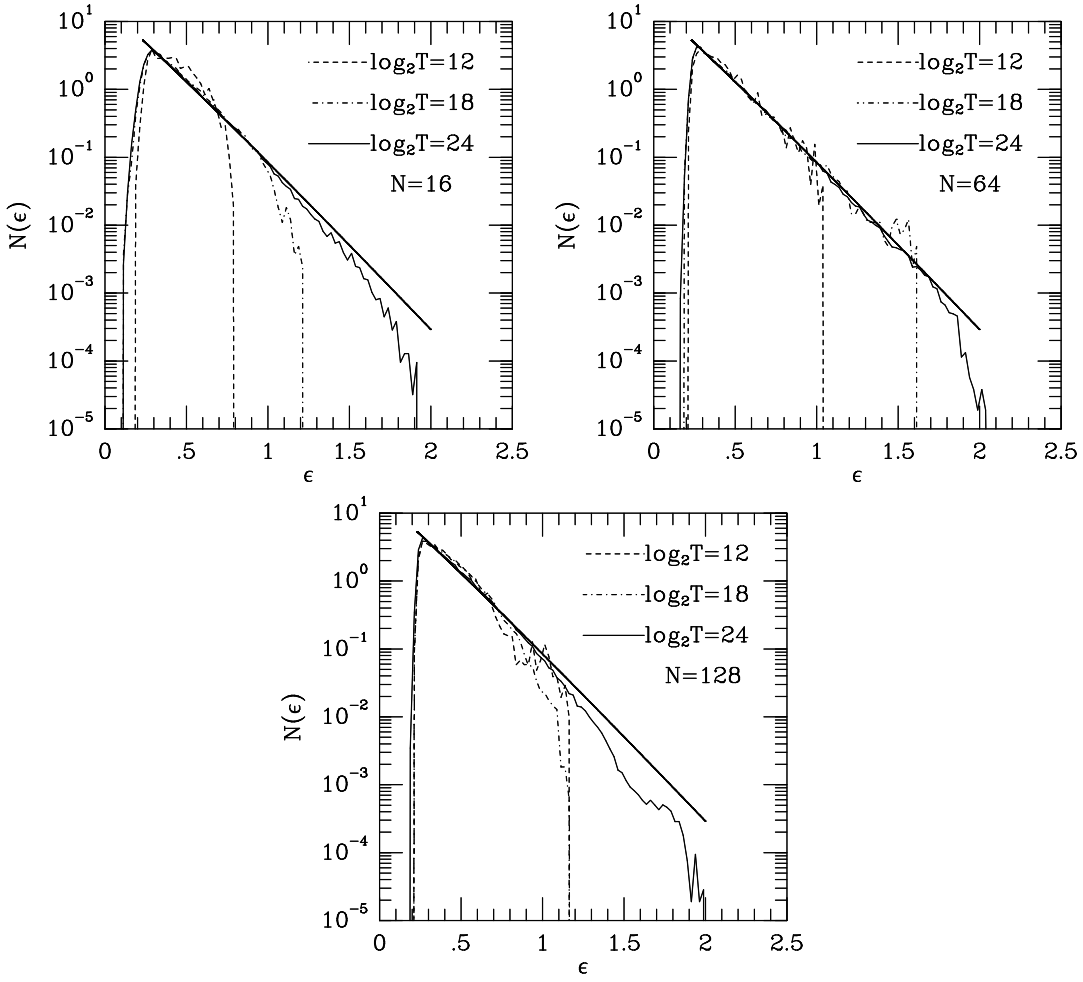

Figure 1: The time-averaged distribution function in the energy space $N(E)$; (a) $N = 16$, (b) $N = 64$, (c) $N = 128$.

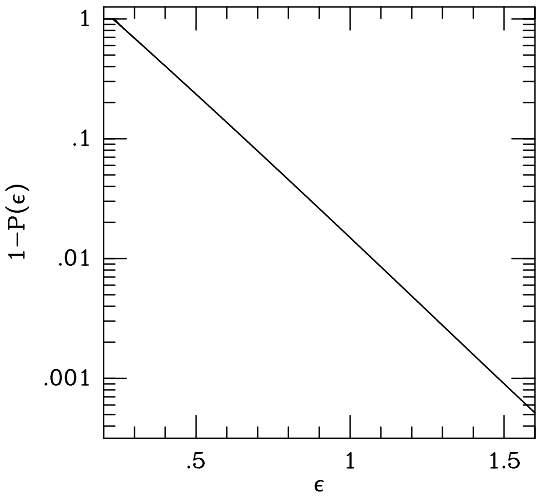

Figure 2: The compliment of the cumulative distribution function $1 - P(e)$ for the thermal equilibrium.

of the relaxation in many studies (see, *e.g.*, Hernquist and Barnes,[1] Hernquist *et al.* [2]), for three-dimensional systems. However, to our knowledge this measure has not been used for the study of the sheet model.

In order to see the dependence of these diffusion coefficients on the energy, we calculated them for intervals of $\Delta\varepsilon = 0.15$. Figure 3 shows the results, for $N = 16, 64$ and 256. The time interval Δt was taken equal to Nt_c. We used smaller values for Δt and confirmed that the choice of Δt has negligible effect if Δt is larger than $10t_c$ and smaller than $4Nt_c$. Time average is taken over the whole simulation period. We can see that both the first- and second-order terms show very strong dependence on the energy of the sheets, and of the order of $1/100N$ for $e \sim 1.5$. Figure 3 suggests that the relaxation timescale grows exponentially as energy grows. This behavior is independent of the value of N.

We can define the relaxation timescale as

$$t_r = \varepsilon^2 / D_2, \tag{10}$$

that is, the timescale in which energy changes significantly. Figure 4 shows this relaxation timescale for different values of N and ε. The relaxation time shows very strong dependence on the energy and the relaxation of high-energy sheets is much slower than that of sheets in lower energies. This is partly because of the dependence of t_r on ε itself. However, as w can see in figure 3, the dependence of the diffusion coefficient is the main reason.

This result resolves the apparent contradiction between the fact that the relaxation timescale is of the order of Nt_c [5] and that the system reaches the true thermal equilibrium only in much longer timescale.[12] It is true that the relaxation timescale is $O(N)$, but the coefficient before N is quite large, in particular for sheets with high energies.

An important question is why the relaxation timescale depends so strongly on the energy. This is provably due to the fact that high-energy sheets have the orbital period significantly longer than the crossing time. Typical sheets have the period comparable to the crossing time, and therefore they are in strong resonance with each other. However, a high-energy sheet has the period longer than the crossing time, and thus it is out of resonance with the rest of the system. Therefore, the coupling between high energy sheets and the rest of the system is much weaker than the coupling between sheets with average energy. This explains why the relaxation of high energy sheets is slow.

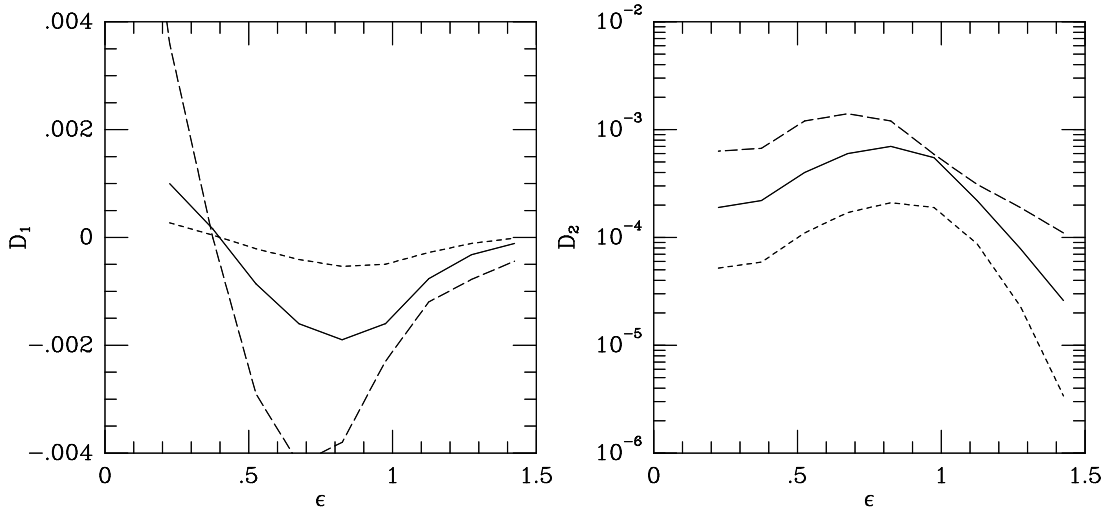

Figure 3: The diffusion coefficients (a) D_1 and (b) D_2 plotted against the energy e for three values of N. Long-dashed, solid, and short-dashed curves are the results for $N = 16$, 64 and 256, respectively.

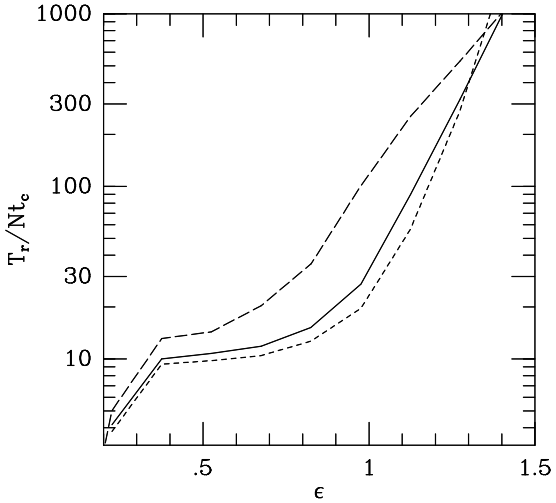

Figure 4: The relaxation time in unit of Nt_c plotted against the energy e for three values of N. Curves have the same meanings as in figure 3

4 Summary and Discussion

In this paper, we studied the thermal relaxation process in one-dimensional self-gravitating systems. We confirmed the result obtained by Tsuchiya *et al.*[12] that the thermal relaxation takes place in the timescale much longer than Nt_c. However, we found that this is simply because the thermal relaxation timescale is much longer than Nt_c. Even for typical sheets, the relaxation timescale is around $10Nt_c$. In order to obtain good statistics, we need to take average over many relaxation times. Moreover, the relaxation time for sheets in the high-energy end of the distribution function is even longer, since the relaxation timescale grows exponentially as the energy grows. Thus, it is not surprising that we have to wait for more than $10^4 Nt_c$ to obtain good statistics.

Does this finding have any theoretical/practical relevance? Theoretically, there is nothing new in our result. What we found is simply that numerical simulation should cover the period much longer than the relaxation timescale to obtain statistical properties of the system, and that the relaxation timescale of a sheet depends on its energy. Both are obvious, but some of the previous studies neglected one or both of the above, and claimed to have found a complex behavior, which, in our view, is just a random walk.

Our finding of the long relaxation time by itself has rather little astrophysical significance, since in the large N limit, the relaxation time is infinite anyway. However, since any numerical simulation suffers some form of numerical relaxation, it is rather important to understand how the relaxation effect changes the system. To illustrate this, we examine the claims by Tsuchiya *et al.*[12] in some detail here.

They argued that the evolution of the mass sheet model proceeds in the following four steps: (1) viliarization, (2) dynamical equilibrium, (3) quasiequilibrium, and (4) thermal equilibrium. According to them, the viliarization timescale is order of t_c, and the energy of each sheet is "conserved" in the dynamical equilibrium phase, which continues up to $t \sim Nt_c$. Then, "microscopic relaxation" takes place in the timescale of $t \sim Nt_c$, where the energy of each sheet is relaxed, but the whole system needs timescale much longer to reach the true equilibrium, because of some complex structure in the phase space.

Our numerical results are in good agreement with those of Tsuchiya *et al.*,[12] but our interpretation is much simpler: First system virializes, and then relaxation proceeds in the timescale of thermal relaxation, which depends on the energy of the individual sheets. Thus, the central region with short relaxation time relaxes to the distribution close to the thermal relaxation in less than $100Nt_c$, but the distribution in the high-energy tail takes much longer to settle. In addition, the small number statistics in the high-energy region makes it necessary to average over many relaxation times to obtain good statistics. In other words, there are no distinction between the "microscopic" and "macroscopic" relaxation, and the evolution of the system is perfectly understood in terms of the standard thermal relaxation.

Acknowledgments

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